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A new approach to the inverse diffraction problem†

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Abstract. In recent studies of the inverse diffraction problem, i.e. the problem of recovering the field distribution in the plane $z = z_1 \geqslant 0$ from a knowledge of the field in an arbitrary plane $z = z_2 > z_1$ in the half-space $z \geqslant 0$ into which the field is propagated, a solution was sought in the form of a linear integral transform. In this paper a different approach is employed and a formal solution is obtained in terms of a differential rather than an integral operator. A useful representation for the differential operator, which is valid for fields whose spatial frequency spectrum is band-limited to a circle whose radius is equal to the wave number of the field, is also given.

1. Introduction

The inverse diffraction problem, i.e. the problem of recovering the field distribution in the plane $z = z_1 \geqslant 0$ from a knowledge of the field in an arbitrary plane $z = z_2 > z_1$ in the half-space $z \geqslant 0$ into which it is propagated, has been considered recently by several authors (Wolf and Shewell 1967, Sherman 1967, Lalor 1968 a, b, Shewell and Wolf 1968). It may be shown to be equivalent to the problem of inverting the well-known Rayleigh diffraction integral

$$U(x_2, y_2, z_2) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} U(x_1, y_1, z_1) \frac{\partial}{\partial z'} \left\{ \frac{\exp(\mathrm{i}kr)}{r} \right\} \bigg|_{z'=z_1} \mathrm{d}x_1 \, \mathrm{d}y_1 \tag{1.1}$$

where $z_2 > z_1$

$$r = \{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z' - z_2)^2\}^{1/2}$$

and the integration is carried out over the plane $z_1 =$ constant. The solution to this problem may formally be expressed in the form

$$U(x_1, y_1, z_1) = \hat{L}(x_1, y_1, z_1; x_2, y_2, z_2) U(x_2, y_2, z_2) \tag{1.2}$$

where $\hat{L}(x_1, y_1\ z_1; x_2, y_2, z_2)$ is a linear operator. Hitherto, a solution has been sought in the form of an integral transform

$$U(x_1, y_1, z_1) = \iint_{-\infty}^{\infty} U(x_2, y_2, z_2) K(x_1, y_1, z_1; x_2, y_2, z_2) \, \mathrm{d}x_2 \, \mathrm{d}y_2$$

with $K(x_1, y_1, z_1; x_2, y_2, z_2)$ being a suitable kernel. This approach has led to a solution of the problem under rather general conditions.

This paper describes an alternative approach which leads to an expression for $\hat{L}$ in (1.2) as a differential rather than an integral operator.

2. Operational calculus‡

Consider the convolution transform

$$f(x, y) = \iint_{-\infty}^{\infty} G(x - s, y - t) \phi(s, t) \, \mathrm{d}s \, \mathrm{d}t \tag{2.1}$$

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‡ It is clear that in this section our formal analysis ignores questions of rigour, particularly those relating to the existence and uniqueness of the inversion process described by equation (2.5) *et seq.* In this connection see Hirschman and Widder (1955) and Jones (1966, § 10.4, § 7.7, § 8.7 *et seq.*).

or in standard notation

$$f(x, y) = G(x, y)*\phi(x, y). \tag{2.2}$$

The convolution theorem of Fourier transform theory states that

$$G(x, y)*\phi(x, y) = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} \tilde{G}(p, q)\tilde{\phi}(p, q) \exp\{i(px+qy)\} \, dp \, dq \tag{2.3}$$

where $\tilde{G}(p, q)$ is the Fourier transform of $G(x, y)$ defined as follows:

$$\tilde{G}(p, q) \equiv \iint\limits_{-\infty}^{\infty} G(x, y) \exp\{-i(px+qy)\} \, dx \, dy \tag{2.4}$$

and $\tilde{\phi}(p, q)$ is the Fourier transform of $\phi(x, y)$, defined in a similar way.

Let us define the function

$$E(p, q) \equiv \{\tilde{G}(p, q)\}^{-1} \tag{2.5}$$

and let us apply to both sides of (2.3) the operator† $\hat{E}(-i\partial/\partial x, -i\partial/\partial y)$. To operate with $\hat{E}(-i\partial/\partial x, -i\partial/\partial y)$ we expand it in a power series in which the derivatives are treated as algebraic quantities, then put $(\partial/\partial x)^n = \partial^n/\partial x^n$ etc. and operate with the derivatives term by term. Upon carrying out this operation and taking the derivatives inside the integral on the right-hand side we obtain

$$\hat{E}\left(-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right) G(x, y)*\phi(x, y)$$
$$= \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} \tilde{G}(p, q)\{\tilde{G}(p, q)\}^{-1}\tilde{\phi}(p, q) \exp\{i(px, qy)\} \, dp \, dq. \tag{2.6}$$

Equation (2.6) may be simplified to give

$$\hat{E}\left(-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right) G(x, y)*\phi(x, y)$$
$$= \left(\frac{1}{2\pi}\right)^2 \iint\limits_{-\infty}^{\infty} \tilde{\phi}(p, q) \exp\{i(px, qy)\} \, dp \, dq$$
$$= \phi(x, y)$$

where the last step is obtained with the help of the Fourier inversion theorem. We have thus formally inverted the convolution transform and obtained a solution in the form

$$\hat{E}\left(-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right) f(x, y) = \phi(x, y).$$

3. The inversion of the Rayleigh transform

The Rayleigh transform (1.1) is a convolution transform of the form (2.1), with

$$U(x_2, y_2, z_2) \equiv f(x, y) \qquad \text{where } x_2 = x, y_2 = y \tag{3.1}$$

$$U(x_1, y_1, z_1) \equiv \phi(s, t) \qquad \text{where } x_1 = s, y_1 = t \tag{3.2}$$

and

$$\frac{1}{2\pi}\frac{\partial}{\partial z'}\frac{\exp[ik\{(x-s)^2+(y-t)^2+(z_2-z')^2\}^{1/2}]}{\{(x-s)^2+(y-t)^2+(z'-z_2)^2\}^{1/2}}\bigg|_{z'=z_1} \equiv G(x-s, y-t). \tag{3.3}$$

† The *operator* $\hat{E}(-i\partial/\partial x, -i\partial/\partial y)$ has the same functional form in terms of $-i\partial/\partial x$ and $-i\partial/\partial y$ as the *function* $E(p, q)$ has in terms of p and q.

The Fourier transform $\tilde{G}(p, q)$ of the function G given by (3.3) may readily be calculated and is found to be (cf. Lalor 1968c)

$$\tilde{G}(p, q) = \exp\{i(z_2 - z_1)(k^2 - p^2 - q^2)^{1/2}\} \tag{3.4}$$

where the positive square root is implied. Thus in this case the operator $\hat{E}$ becomes

$$\hat{E}\left(-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right) \equiv \exp\left\{-i(z_2 - z_1)\left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{1/2}\right\}. \tag{3.5}$$

The solution to the inverse diffraction problem equation (1.2) may therefore be expressed in the form

$$\exp\left\{-i(z_2 - z_1)\left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{1/2}\right\} U(x, y, z_2) = U(x, y, z_1), \qquad \text{for } z_2 > z_1. \tag{3.6}$$

It is of interest to note that the solution to the original diffraction problem may be expressed in a similar form. For, if we operate on both sides of equation (3.6) with the operator

$$\exp\left\{+i(z_2 - z_1)\left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{1/2}\right\}$$

we obtain

$$U(x, y, z_2) = \exp\left\{i(z_2 - z_1)\left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{1/2}\right\} U(x, y, z_1). \tag{3.7}$$

This is an alternative form of the solution to the direct diffraction problem, given by the Rayleigh formula equation (1.1).

Equation (3.7) appears to have been derived first by Bremmer (1952, 1959), using a different approach. However, the solution for the inverse diffraction problem in the form (3.6) appears to be new.

Equations (3.6) and (3.7) express the solutions in a rather formal way. We will show, in the next section, that for a wide class of fields, the above solutions may be expressed in a more explicit form.

4. The non-evanescent wave field

Under rather general conditions it is possible to represent the field as an angular spectrum of plane waves (see, for instance, Bouwkamp 1954, Lalor 1968c):

$$U(x, y, z) = \left(\frac{k}{2\pi}\right)^2 \iint_{-\infty}^{\infty} A(p, q) \exp\{ik(px + qy + mz)\} \, dp \, dq, \qquad \text{for } z \geqslant 0 \tag{4.1}$$

where

$$\begin{aligned} m &= (1 - p^2 - q^2)^{1/2} && \text{if} \quad p^2 + q^2 < 1 \\ &= +i(p^2 + q^2 - 1)^{1/2} && \text{if} \quad p^2 + q^2 \geqslant 1 \end{aligned}$$

and $A(p, q)$ in the Fourier transform of the field in the plane $z = 0$, i.e.

$$A(p, q) = \iint_{-\infty}^{\infty} U(x, y, 0) \exp\{-ik(px + qy)\} \, dx \, dy.$$

Real values of m are seen to be associated with homogeneous plane waves and imaginary values with evanescent waves. In this section we shall consider fields which contain no evanescent waves. We will call such fields *non-evanescent wave fields*. Sherman (1968a, b) and Shewell and Wolf (to be published) have discovered some interesting properties of these fields. Sherman calls them 'source-free fields' for such fields have no sources anywhere (including infinity). It is readily seen that the non-evanescent wave field contains no spatial periodicities (in the sense of Shewell and Wolf 1968), smaller than the wavelength of the radiation, in any plane $z = z_1$. However, since the evanescent waves, which carry information about details smaller than the wavelength, are rapidly damped out, particularly at optical frequencies, one might expect that the non-evanescent wave field would provide a good approximation to the total field for most cases of practical interest. We see from (4.1) that a non-evanescent wave field may be represented in the form

$$U_{\mathrm{NE}}(x, y, z) = \left(\frac{k}{2\pi}\right)^2 \iint_{p^2+q^2<1} A(p, q) \exp[ik\{px+qy+(1-p^2-q^2)^{1/2}z\}]\, \mathrm{d}p\, \mathrm{d}q. \tag{4.2}$$

To find a representation for the differential operators in (3.6) and (3.7) we make use of the expansions (cf. Watson 1944†)

$$\exp\{\mathrm{i}z(a^2-b^2)^{1/2}\} = (\tfrac{1}{2}\pi az)^{1/2} \sum_{n=0}^{\infty} \left\{\frac{\mathrm{H}^{(1)}_{n-\frac{1}{2}}(az)}{n!}\right\}\left(\frac{b^2z}{2a}\right)^n \tag{4.3}$$

and

$$\exp\{-\mathrm{i}z(a^2-b^2)^{1/2}\} = (\tfrac{1}{2}\pi az)^{1/2} \sum_{n=0}^{\infty} \left\{\frac{\mathrm{H}^{(2)}_{n-\frac{1}{2}}(az)}{n!}\right\}\left(\frac{b^2z}{2a}\right)^n.$$

These expansions are valid provided that $|b| < |a|$. $\mathrm{H}^{(1)}_{n-\frac{1}{2}}$ and $\mathrm{H}^{(2)}_{n-\frac{1}{2}}$ are the Hankel functions of the first and second kind respectively, of order $n-\frac{1}{2}$.

Thus we may rewrite equations (3.6) and (3.7) as

$$U(x, y, z_1) = \{\tfrac{1}{2}\pi k(z_2-z_1)\}^{1/2} \sum_{n=0}^{\infty} \left\{\mathrm{H}^{(2)}_{n-\frac{1}{2}}\left(\frac{k(z_2-z_1)}{n!}\right)\right\}\left(\frac{z_1-z_2}{2k}\right)^n \Delta_2{}^n U(x, y, z_2) \tag{4.4}$$

and

$$U(x, y, z_2) = \{\tfrac{1}{2}\pi k(z_2-z_1)\}^{1/2} \sum_{n=0}^{\infty} \left\{\mathrm{H}^{(1)}_{n-\frac{1}{2}}\left(\frac{k(z_2-z_1)}{n!}\right)\right\}\left(\frac{z_1-z_2}{2k}\right)^n \Delta_2{}^n U(x, y, z_1) \tag{4.5}$$

respectively, where we have substituted $\Delta_2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ for $-b^2$ in equation (4.3). $\Delta_2{}^n$ is, of course, to be interpreted as the operator resulting from the application of the operator Δ_2, n times in succession. The correctness of equations (4.4) and (4.5) may readily be verified for wave fields that are non-evanescent. For this purpose one expresses the fields in the form (4.2) and takes the operator inside the integral sign on the right-hand side. Further, one makes use of the fact that since the x and y dependence in the integrand is entirely in the exponent, the result of the operation is to replace Δ_2 by $-k^2(p^2+q^2)$, whenever it appears. Since‡ $p^2+q^2 < 1$ the series appearing in the integrand may be summed and gives $\mp ik(1-p^2-q^2)^{1/2}(z_2-z_1)$, the negative sign referring to equation (4.4) and the positive sign to equation (4.5).

Equation (4.5) for direct diffraction was first obtained by Bremmer (1952) and recently derived in a different way by Sherman (1968 b). Equation (4.4), which represents a solution to the inverse diffraction problem, appears to be new.

The remarkable symmetry between the solutions to the direct and inverse diffraction problems for non-evanescent fields, given by our equations (4.5) and (4.4), is just as apparent in the integral transform approach of Wolf and Shewell (1967). The reciprocity theorem of Shewell and Wolf (1968) for non-evanescent wave fields may immediately be verified from equations (4.4) and (4.5).

† Equations (4.3) may be deduced in a straightforward manner from (3) and (4) of § 5.22 of this reference.

‡ This corresponds to the condition $|b| < |a|$ of equation (4.3).

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